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Two-Wheeled Robotic Mobility Platform

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Abstract— This paper presents the results on the modelling and control of a two wheeled robotic mobility platform that achieves automatic self balance detecting the weight shift of the operator. This vehicle is principally a self balancing machine whose wheels share a common axis. Self balance is achieved by accelerating the body enabling pseudo forces to act upon it thus bringing the body to its mean position. We have employed Newton's laws of motion to determine the acceleration required to achieve self balance. This condition of achieving self balance has been put to use by us to achieve locomotion. For oscillation-free motion a new concept called cut-off angle has also been introduced. This machine has numerous advantages which make it an asset considering the problems faced by present day commuters especially on crowded Indian roads where pollution, fuel crisis and traffic congestion are the major hitches. Being a battery operated vehicle there is zero emission. Moreover, as both wheels share a common axis, the turning radius can be easily minimised to zero. Apart from this a new method to calculate the net moment of inertia of vehicles has also been discussed in this paper.

Keywords— Segway; pseudoforce; moment of inertia; cut-off angle

1. Introduction

The Conquest of robots in human's life is in different ways like guard robots, services robots, fire-fighter robots, entertaining robots and human robots. There is an immense research going on making them cheap, efficient and reliable. An important class of autonomous wheel mobile robots is dual wheel upright self-balancing robots. Their movement is usually flexible. Dual wheel Robots has two wheels and a vertical body frame on which all circuitry is placed. Its mechanism is like a human that balances. It adjusts its position, when it is about to fall forward or backward to avoid the instability. Unlike conventional mobile robots, dual wheel self-balancing robots bring practical advantages as follows:

- Movement on zero radius curve
- High tolerance to impulsive force
- Small foot print help to move on dangerous places
- Greater stability over slopes.

The RMP is a hardware component that has been developed under a program whose objectives focus on the development of software: DARPA, MARS program. The reason for this is that the validation of autonomous capabilities always requires testing on physical robots in real world environments, and this requires physical hardware: robotic vehicles, sensors, and computers. Too often the cost of these components, coupled with the cost of performing experimentation and testing, diverts both financial resources and management attention from the primary program goals. "Traditional" platforms for mobile robot research (which include oversized "toy cars" such as the Pioneer and large upright "trash can" robots such as the Nomad) have been limited in terms of both payload capacity and mobility characteristics, including speed.

Fig.1 Traditional platforms for mobile robot research

Abbreviations and Acronyms

RMP- Robotic Mobility Platform
DARPA- Defense Advanced Research Projects Agency
MARS- Mobile Autonomous Robot Software

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2. Principle

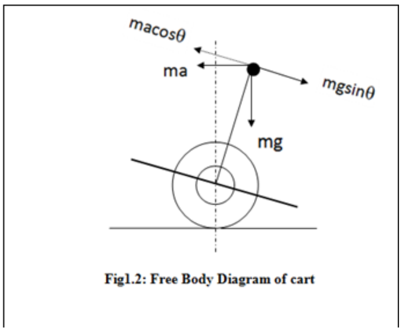

Fig. 2.1: Free Body Diagram of Cart

The analysis can be done using Newton's Laws of Motion. The free body diagram of the vehicle while in motion is shown in the figure above. As the vehicle wheel accelerates the centre of gravity of the system which is a combination of the cart and the person driving it experiences an imaginary force known as pseudo force in a direction opposite to the direction of acceleration. This force is the balancing force which brings the tilted platform to its mean position. From the free body diagram we can derive the following set of equations.

$$m * a * \cos\theta = m * g * \sin\theta$$

Therefore,

$$a = g * \tan\theta \qquad ...(2.1)$$

According to Lagrangian equation for an inverted pendulum,

$$l * \alpha - g * \sin\theta = \ddot{x} * \cos\theta \qquad . \quad ..(2.2)$$

Now, if the platform remains in neutral equilibrium in its tilted position, the value of α i.e the angular acceleration of the platform, is zero. Thus the equation can be further simplified as

$$a = g * \tan\theta \qquad ...(2.3)$$

Hence, the equation (2.1) is well in agreement with the Lagrangian equation (2.2),

3. Analysis

If the cart is given an acceleration which is equal to the value obtained from the above equation the platform will stay in equilibrium in its tilted position. In order to bring the platform back to its original position the acceleration given to the cart should be slightly greater than the calculated acceleration. Consider the figure 2.1 shown above.

Let I_p be the moment of inertia of the platform about the wheel axis and ω_p and α_p be the instantaneous angular velocity and instantaneous angular acceleration of the platform when rotating about the wheel axis. Let h be the height of the centre of gravity from the wheel axis and 'm' be the mass of the loaded system excluding the mass of the wheels. From the free body diagram

$$I_p * \alpha_p = m * a * \cos\theta - m * g * \sin\theta * h$$
$$= m * h * \cos\theta * (a - g * \tan\theta)$$
$$= m * h * \cos\theta * a_e$$

Therefore $\alpha_p = {}^{m * h}/_{I_p} * \cos\theta * a_e \qquad ... (3.1)$

Where a_e , can be called excessive acceleration.

The value of excessive acceleration should be a function of the angle of tilt so that the net acceleration is reduced to zero when the platform comes to its normal horizontal position.

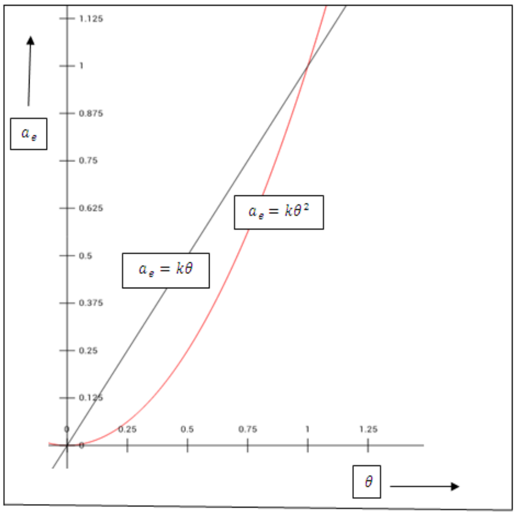

Fig 3.1: Graph of excessive acceleration vs angle of tilt

Now, the function should be such that the excessive acceleration is low for small angles of tilt and high for larger angles of tilt. A parabolic function can provide these desired characteristics. This is clear from fig. 3.1. Hence to minimise jerks the excessive acceleration is kept directly proportional to the square of the angle of tilt.

Therefore, $a_e \propto \theta^2$

$$a_e = k * \theta^2 \qquad .... (3.2)$$

Where, k is the constant of proportionality whose best value is chosen in the succeeding sections.
Consider fig. 2.1,
Let θ_0 be initial angle of tilt. It is assumed that time't'=0 at this instant.

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Also the angular velocity of platform ω_p=0
Consider Newton's equation of motion,
$v^2 = u^2 + 2as$
Let dω (infinitesimally small) be the corresponding change in the angular velocity for the angular displacement of dθ of the platform such that the angular acceleration α_p can be assumed to be constant for that duration.

Applying above equation of motion for angular motion we can write,
$(\omega + d\omega)^2 = \omega^2 + 2\alpha_p * d\theta$
$\omega^2 + 2\omega * d\omega + d\omega^2 = \omega^2 + 2\alpha_p * d\theta$
The $d\omega^2$ is very small and can be neglected.
Hence, the above equation can be written as,

$$\omega * d\omega = \alpha_p * d\theta \; \ldots\ldots (3.3)$$

But, $$\alpha_p = {}^{m*h}/_{I_p} * cos\theta * a_s$$

and, $$a_s = k * \theta^2$$

Substituting, the values of α_p and a_s in equation (3.3),
$$\omega * d\omega = {}^{m*h}/_{I_p} * cos\theta * k * \theta^2 * (-d\theta)$$
Integrating both sides,
$$\int \omega * d\omega = -\int {}^{m*h}/_{I_p} * cos\theta * k * \theta^2 * d\theta$$
$$\frac{\omega^2}{2} = -\frac{m*h*k}{I_p}\left((\theta^2 - 2)\sin\theta + 2*\theta*\cos\theta\right) + c$$

$$When\ \theta = \theta_0, \omega = 0$$

$$0 = -\frac{m*h*k}{I_p}\left((\theta_0^2 - 2)\sin\theta_0 + 2\theta_0\cos\theta_0\right) + c$$

$$c = \frac{m*h*k}{I_p}((\theta_0 - 2)sin\theta_0 + 2\theta_0 cos\theta_0)$$

$$\frac{\omega^2}{2} = -\frac{m*h*k}{I_p}\left((\theta^2 sin\theta - \theta_0^2 sin\theta_0) - 2(sin\theta - sin\theta_0) + 2(\theta cos\theta - \theta_0 cos\theta_0)\right) \quad \ldots (3.4)$$

From the above equation, the angular velocity of platform about its wheel axis can be determined for any angle of tilt θ.
To make the platform stationary when it reaches the horizontal position it is required to switch off the system before it reaches the equilibrium position. The angle at which the system should be switched off will vary with the initial angle of tilt. This angle at which the system is switched off is called cut-off angle and is denoted by θ_c.

4. Nature of Motion

Let ω_f be the angular velocity of platform when it reaches the cut-off angle. Free Body Diagram of the system at cut-off angle is shown in figure 4.1

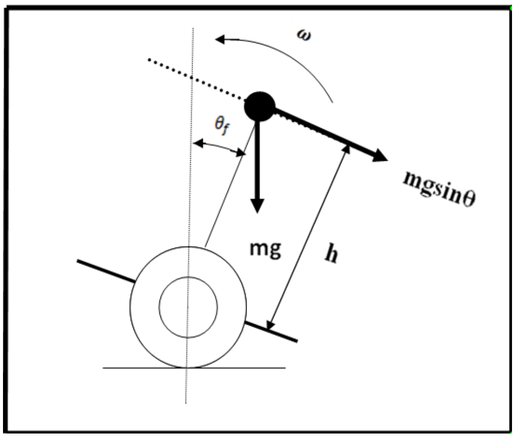

Fig 4.1: FBD of system at cut-off angle

The angular acceleration of the cart can be stated as:
$$\alpha_f = \frac{-m*g*sin\theta*h}{I_p}$$
Using Newton's law of motion stated before, we know,
$v^2 = u^2 + 2as$
Therefore, in terms of angular motion of the cart, we can represent Newton's law as follows:
$$(\omega - d\omega)^2 = \omega^2 - 2*\frac{m*g*h*sin\theta}{I_p}d\theta$$
Thus, solving the equation above and also assuming that $d\omega^2$ is negligible, we obtain the following simplification:
$$\omega^2 - 2\omega d\omega = \omega^2 - 2*\frac{m*g*h*sin\theta}{I_p}d\theta$$
$$\therefore \int \omega d\omega = \int \frac{m*g*h*sin\theta}{I_p}d\theta$$
$$\frac{\omega^2}{2} = \frac{-m*g*h*cos\theta}{I_p} + c \quad \ldots (4.1)$$
However, to determine the value of constant c we apply the boundary condition i.e. when $\theta = 0$; $\omega = 0$
$$\therefore 0 = \frac{-m*g*h}{I_p} + c$$
$$c = \frac{m*g*h}{I_p} \quad \ldots (4.2)$$

Now comparing equations (4.1) and (4.2) we obtain,
$$\frac{\omega^2}{2} = \frac{-m*g*h*cos\theta}{I_p} + \frac{m*g*h}{I_p}$$
$$\therefore \frac{\omega^2}{2} = \frac{m*g*h*(1-cos\theta)}{I_p} \quad \ldots (4.3)$$

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The nature of motion of the platform from the initial angle of tilt to the horizontal position is governed by equations Applying the boundary condition such that, when $\theta = \theta_c, \omega = \omega_c$

$$\therefore \frac{\omega_c^2}{2} = \frac{m*g*h}{I_p}*(1-cos\theta_c) \qquad \dots(4.4)$$

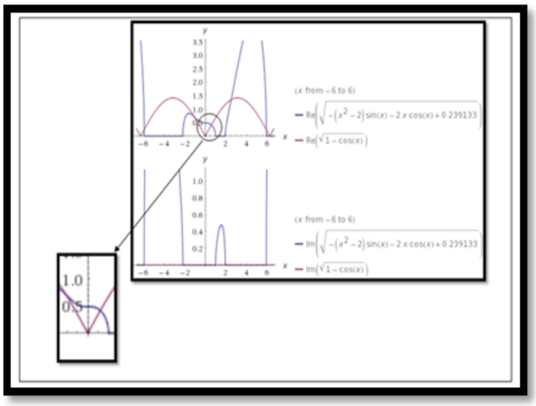

Fig. 4.2: Graph of angular velocity vs angle of tilt

The graphs of equations 3.4(blue) and 4.4(purple) are shown in the figure 4.2. The graph has been generated using the online application created by Wolfram Alpha. The point of intersection of the two graphs denotes the cut-off angle. The angular velocity of the platform traces curve 1(blue curve) up to cut-off angle and then traces curve 2(purp le curve) till it reaches zero angle of tilt and zero angular velocity. To determine the value of cut-off angle equation 3.4 and equation 4.4 should be solved.

$$-\frac{m*h*k}{I_p}\left((\theta^2 sin\theta - \theta_0^2 sin\theta_0) - 2(sin\theta - sin\theta_0) + 2(\theta cos\theta - \theta_0 cos\theta_0)\right) = \frac{m*g*h}{I_p}*(1-cos\theta)$$

Now, denoting the varia
ble θ by x and constant θ_0 by 'p' the above equation can be simplified as

$$(x^2-2)*\sin(x) + \left(2*x-\frac{g}{k}\right)*\cos(x)$$

$$= \left[(p^2-2)*\sin(p) + 2*p*\cos(p) - \frac{g}{k}\right]$$

These two equations can be solved by using numerical methods only. Solving them using any equation solving software we get the following roots at the corresponding angles:

Table.1: Angle of tilt with corresponding Cut-off Angle

ANGLE(°)	*CUT-OFF ANGLE(°)*
1	0.024337
2	0.068859
3	0.126482
4	0.194662
5	0.271937
6	0.357283
7	0.449944
8	0.549334
9	0.654959
10	0.766405
11	0.883311
12	1.005361
13	1.132271
14	1.263778
15	1.399646
16	1.539650
17	1.683583
18	1.831249
19	1.982459
20	2.137036

Thus programming a micro-controller to solve this equation and sending the appropriate signal to the motor drivers of the vehicle can result in an oscillation-free motion.

5. Differential Acceleration while Negotiating a Curve

The velocity and acceleration of the inner and outer wheels while negotiating a turn with an angle of inclination 'θ' has been derived referring the figure 5.1:

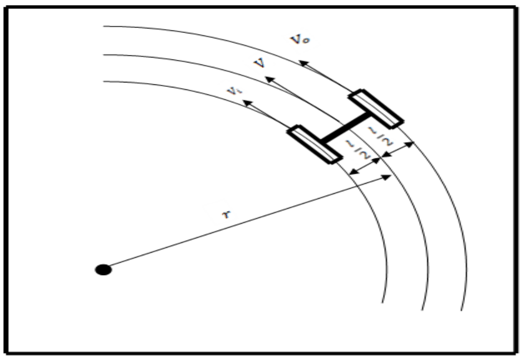

Fig 5.1: Representation of velocity of wheels while negotiating a turn

Now, as the angular velocity at every point on the cart is the same, we can assure that the angular velocities of the

inner wheel, the centre of mass and the outer wheels are the same which can be denoted using the formula:
$V = r\omega$

Where,
V is the linear velocity of the component considered.
r is the turning radius of the component of the cart
ω is the angular velocity of the cart while steering.

Therefore,
$$\frac{V_i}{r-\frac{l}{2}} = \frac{V}{r} = \frac{V_o}{r+\frac{l}{2}} = \omega \quad \dots (5.1)$$

Where
V is the linear velocity of the centre of mass which is assumed to be at the centre of the cart.
V_i is the linear velocity of the inner wheel.
V_o is the linear velocity of the outer wheel.
In general, we can write:

$$\frac{V}{r} = \frac{V_x}{r+x} \quad \dots. \left[\frac{-l}{2} \le x \le \frac{l}{2}\right]$$
Where, 'x' is the distance of the point considered from the centre of mass of the system.
By differentiating the expression (5.1) w.r.t time't', we get the acceleration of the components.

$$\frac{\frac{dV_i}{dt}}{r-\frac{l}{2}} = \frac{\frac{dV}{dt}}{r} = \frac{\frac{dV_o}{dt}}{r+\frac{l}{2}} \quad \dots (5.2)$$

But for an angle of inclination $'\theta'$,
$$\frac{dV}{dt} = acceleration\ 'a' = gtan\theta$$

Acceleration of inner wheel:

$$\frac{\frac{dV_i}{dt}}{r-\frac{l}{2}} = \frac{\frac{dV}{dt}}{r}$$
$$\therefore \frac{a_i}{r-\frac{l}{2}} = \frac{gtan\theta}{r}$$
$$\therefore a_i = \frac{gtan\theta}{r} * \left(r-\frac{l}{2}\right) \quad \dots (5.3)$$

Acceleration of outer wheel:

$$\frac{\frac{dV_o}{dt}}{r+\frac{l}{2}} = \frac{\frac{dV}{dt}}{r}$$
$$\therefore \frac{a_o}{r+\frac{l}{2}} = \frac{gtan\theta}{r}$$

$$\therefore a_o = \frac{gtan\theta}{r} * \left(r+\frac{l}{2}\right) \quad \dots (5.4)$$
When the angle of inclination of the cart tends to zero, the system will not experience any acceleration and will move at constant velocity. This condition of constant velocity of inner and outer wheels can be denoted as:
$$V_i = \frac{V}{r} * \left(r-\frac{l}{2}\right) \quad \dots (5.5)$$
$$V_o = \frac{V}{r} * \left(r+\frac{l}{2}\right) \quad \dots (5.6)$$

6. Effective Moment of Inertia of the Vehicle

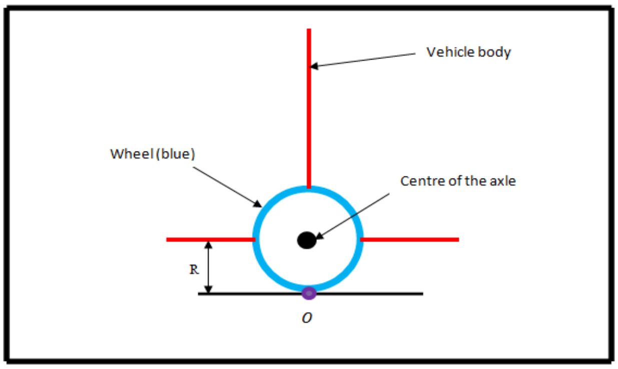

Fig. 6.1: Detailed view of the system

The figure 6.1 gives us a detailed view of the system. We separate the system into the following three parts:
- Vehicle body (red)
- Wheel (blue)
- Centre of the axle(black)

Now for any vehicle undergoing translation, it can be assumed that the body rotates about the axis passing through the point of contact between the wheel and the ground. Every point on the vehicle body except for the rotating member that is the wheel has the same linear velocity which is equal to the linear velocity of the wheel's centre, i.e.
$v = R * \omega$
Where,
$v = linear\ velocity\ of\ the\ point$
$R = distance\ of\ the\ point\ from\ the\ point\ of\ contact\ (which\ in\ this\ case\ is\ the\ radius\ of\ the\ wheel)$
$\omega_n = angular\ velocity\ of\ the\ system\ rotating\ about\ the\ point\ O$

This is assumed so because every part of the vehicle body has just one angular velocity that is the angular velocity with which it is assumed to be rotating about the

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point of contact. Hence we can assume the whole vehicle body to be concentrated at a point at the centre of the axle depicted in red in fig.6.2. That is the point is at a distance $'R'$ from the point of contact. Now, we assume the cumulative mass of the vehicle body to be M_b. Hence, the moment of inertia due to this concentrated mass will be:

$$I_b = M_b * R^2$$

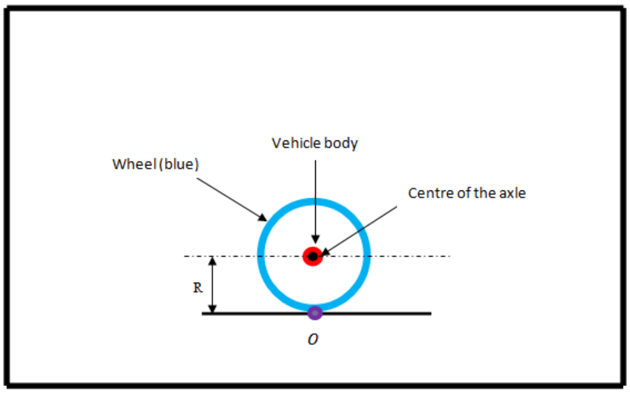

Fig. 6.2: Representation of all rotating members of the system

Every point on the wheel will have a different linear velocity because as we move farther away from the point of contact, the linear velocity increases as per the formula stated above. Apart from the angular velocity ω_n it also has the angular velocity ω_w with which it rotates about the wheel axle. Hence the wheels having mass M_w are considered as a separate entity. Hence, while calculating the moment of inertia of the system we consider the moment of inertia of the wheels separately. This can be determined by using parallel axis theorem, i.e.)

$$I_w = M_w * R^2 + \frac{M_w R^2}{2}$$

Therefore,

$$I_w = \frac{3}{2} * M_w * R^2$$

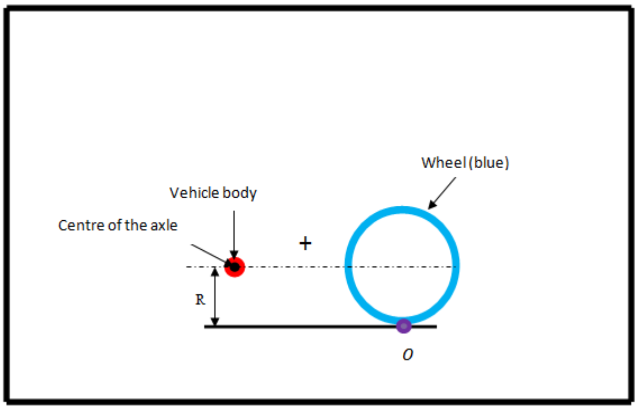

Fig 6.3: Resolution of rotating members of the system

Now, in order to determine the moment of inertia of the system, we obtain the summation of the moments of the rotating and the non-rotating members of the system.
Therefore,

$$I_s = I_{rotating} + I_{non-rotating}$$
$$\therefore I_s = I_w + I_b$$
$$\therefore I_s = \frac{3}{2} * M_w * R^2 + M_b * R^2$$

This equation shows the moment of inertia of the system shown in the figure 6.3. Hence, this derivation can be used to determine the moment of inertia of the system. This is an effective method to calculate the inertia by classifying the components within the system into rotating and non-rotating and accordingly carrying out the calculations. This method has been employed by us in order to determine the torque requirements to decide the motors.

7. Acceleration Control of Motor

According to the electromagnetic induction theories, force acting on a current carrying conductor can be expressed as:

$$F = B * I * l * \sin\theta$$

Where,

$F = Force\ acting\ on\ a\ current\ carrying\ conductor\ (motor\ windings)$
$B = Magnetic\ field$
$I = Current\ passed\ through\ the\ coil$
$l = Length\ of\ the\ current\ carrying\ conductor$
$\theta = Angle\ between\ the\ magnetic\ field\ and\ the\ plane\ of\ the\ coils.$

Now for a motor with coil radius 'r', we can obtain the torque (T) equation as follows:

$$T = F * r$$
$$= B * I * l * \sin\theta * r$$

As the magnetic field inside the motor is radial, the value of θ is $90°$
Hence the value of $\sin\theta$ can be approximated to 1 at every position of the coil. Hence, the torque equation is modifies into:

$$T = B * I * l * r$$

Now, since the values of B, l and r for a given motor remains constant, we can consider the term $B * l * r$ as a constant K_t.
Therefore,

$$T = K_t * I \quad ...$$

However,
By Ohm's law, we know that for a given conductor, the voltage V, current I and resistance R are related as follows:

$$I = V/R$$

Therefore, substituting in the equation above,

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$$T = K_t * \frac{V_{net}}{R} \quad \dots (7.1)$$

The net voltage across the armature $V_{net} = V - E$ ...

Where,

$V = Supply\ voltage$

$E = Back\ EMF$

Back EMF produced in a motor is directly proportional to its angular velocity. In other words,

$$E = K_e * \omega \quad \dots (7.2)$$

Where,

$K_e = Constant\ of\ proportionality$

$\omega = Angular\ velocity$

From equations 7.1 and 7.2, we get,

$$T = K_t * \frac{V - K_e * \omega}{R} \quad \dots (7.3)$$

Motor produces maximum torque when its back EMF is zero, that is, its angular velocity is zero. This maximum torque is known as the 'stall torque' denoted by T_s.

Therefore, substituting the value of ω in the equation , we obtain the stall torque.

$T_s = K_t * V/R$

Thus,

$K_t = R * T_s/V$

The values of the stall torque (T_s) and coil resistance (R) can be obtained from the motor's datasheet.

The final angular speed attained at no load is the maximum angular speed with which the motor can rotate with. This maximum angular speed is denoted by ω_f.

Now at zero load,

$T = T_{if}$

Where,

$T_{if}\ denotes\ the\ torque\ required\ to\ overcome\ the\ internal\ friction\ within\ the\ motor$

Substituting the value of ω_f and T_{if} in the equation (7.3) ,

$$T_{if} = R * \frac{T_s}{V} * (V - K_e * \omega_f)/R$$

Thus,

$$T_{if} = \frac{T_s}{V} * (V - K_e * \omega_f)$$

$$T_{if}/T_s = 1 - K_e * \omega_f/V$$

$$K_e * \frac{\omega_f}{V} = 1 - T_{if}/T_s$$

Therefore,

$$K_e = \frac{V}{\omega_f} * (1 - T_{if}/T_s) \quad \dots (7.4)$$

However, as the internal friction within a motor is negligible, the value of T_{if} can be approximated to zero. Therefore, the equation is modified as:

$$K_e = \frac{V}{\omega_f}$$

$$V_{in} = \frac{I * g * \frac{\tan\theta}{r} + T_f}{K_t} * R + K_e * \omega$$

Substituting values of K_t and K_e

$$V_{in} = \frac{I * g * \tan\theta + r * T_f}{r * (T_s * \frac{R}{V})} * R + V/\omega_f * \omega$$

$$V_{in} = \frac{I * g * \tan\theta/r + T_f}{T_s/V} + V * \omega/\omega_f$$

Thus a direct relation between angle of tilt and input voltage can be obtained.

8. Safety System

Since automatic self balance is achieved by providing suitable acceleration or deceleration the velocity of the cart may increase or decrease during the self balance process. Moreover during locomotion if the operator continuously leans forward so as to reach his/her destination quickly the velocity of the cart will go on increasing due to continuous acceleration. But there is a limit to the maximum velocity that can be provided by the electric motors. This value of maximum rotation per minute is usually given in the motor datasheet. Thus the RPM of the motor shaft cannot be accelerated to a value greater than the maximum achievable value given in the datasheet.

If the velocity of the cart goes on increasing due to continuous acceleration and the shaft reaches a speed which is the equal to the maximum speed that can be provided by the motor the cart will no longer be able to accelerate further. Thus the balancing mechanism for forward tilt will freeze since the cart has achieved the maximum forward velocity. At the same time the mechanism will be able to respond to backward tilt as the cart can still be decelerated till it reaches the maximum backward velocity. Such a situation can be dangerous as failure of the balancing system can cause serious accidents. Hence the operator must be signalled by the cart when the cart approaches this maximum velocity. This signal can be in the form of a blinking light or an alarm. On receiving the signal the operator must decelerate by providing a backward tilt to the cart. Thus, in this manner, the velocity of the cart can be limited below the maximum value.

If N_{max} is the maximum RPM of the motor then the caution signal should be given to the operator at a speed lesser than N_{max}. Let this speed be denoted N_{cr} which stands for critical speed. If N_{max} is known the value of N_{cr} can be calculated as follows:

Let us assume that the motor shaft reaches the speed N_{cr} when it is tilted to the maximum angle which is 20° in this case.

From equation 3.4 we have,

$$\frac{\omega_p^2}{2} = -\frac{m * h * k}{I_p}\left((\theta^2 \sin\theta - \theta_0^2 \sin\theta_0) - 2(\sin\theta - \sin\theta_0) + 2(\theta\cos\theta - \theta_0\cos\theta_0)\right)$$

Where ω_p is the angular velocity of the platform

Substituting $\theta_0 = 0.3490rad$ we get,

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$$\omega_p = \frac{\sqrt{\frac{2mhk}{I_p}} *}{\sqrt{-\left(\begin{array}{c}(\theta^2 sin\theta - 0.041674) - 2(sin\theta - 0.34202) \\ +2(\theta cos\theta - 0.328015)\end{array}\right)}}$$
…(8.1)

But $\omega_p = -\frac{d\theta_p}{dt}$, where θ_p is the angle of tilt.

Substituting in equation 8.1 we get,

$$\frac{d\theta_p}{dt} = \frac{-\sqrt{\frac{2mhk}{I_p}} *}{\sqrt{-\left(\begin{array}{c}(\theta^2 sin\theta - 0.041674) - \\ 2(sin\theta - 0.34202) + 2(\theta cos\theta - 0.328015)\end{array}\right)}}$$
…(8.2)

Now, the acceleration given to the cart is

$$a = g * tan\theta + k * \theta^2$$

$$\frac{dv}{dt} = g * tan\theta + k * \theta^2$$

$$\frac{dv}{d\theta} * \frac{d\theta}{dt} = g * tan\theta + k * \theta^2 \qquad \text{…(8.3)}$$

Substituting value of $\frac{d\theta}{dt}$ in (8.3)

$$\frac{dv}{d\theta} = -\frac{g * tan\theta + k * \theta^2}{\sqrt{\frac{2mhk}{I_p}} * \sqrt{-\left(\begin{array}{c}(\theta^2 sin\theta - 0.041674) \\ -2(sin\theta - 0.34202) + \\ 2(\theta cos\theta - 0.328015)\end{array}\right)}}$$

Substituting the values of g as and 9.81 and converting the above equation to variable separable form we get

$$\int dv = \int -\frac{9.81 * tan\theta + 0.5 * \theta^2}{\sqrt{\frac{2mhk}{I_p}} * \sqrt{-\left(\begin{array}{c}(\theta^2 sin\theta - 0.041674) \\ -2(sin\theta - 0.34202) + \\ 2(\theta cos\theta - 0.328015)\end{array}\right)}} d\theta$$

When the platform rotates from maximum angle of tilt to cut-off angle the velocity of the cart changes from critical velocity (V_{cr}) to maximum achievable velocity (V_{max}). Applying these limits to the above integral we get,

$$V_{max} - V_{cr} = \int_{0.3490}^{0.0373} \frac{g * tan\theta + k * \theta^2}{\sqrt{\frac{2mhk}{I_p}} * \sqrt{-\left(\begin{array}{c}(\theta^2 sin\theta - 0.041674) \\ -2(sin\theta - 0.34202) + \\ 2(\theta cos\theta - 0.328015)\end{array}\right)}} d\theta$$

The integral given above can be solved using numerical methods. Here we use the online solver Wolfram Alpha. The screen shot of the solution as obtained by the solver is given below.

$$\int_{0.349}^{0.0373} -\left(9.81 \tan(\theta) + 0.5\,\theta^2\right) \Big/ \left(\sqrt{\left(-\left((\theta^2 \sin(\theta) - 0.041674) - 2\,(\sin(\theta) - 0.34202) + 2\,(\theta \cos(\theta) - 0.328015)\right)\right)}\right) d\theta = 9.0096$$

Therefore,

$$V_{max} - V_{cr} = \frac{9}{\sqrt{\frac{2mhk}{I_p}}}$$

Or $$V_{cr} = V_{max} - \frac{9}{\sqrt{\frac{2mhk}{I_p}}} \qquad \text{…(8.4)}$$

But $V_{max} = \frac{2\pi N_{max}}{60}$ and $V_{cr} = \frac{2\pi N_{cr}}{60}$

Substituting these values in (8.4) we get

$$N_{cr} = \frac{60}{2\pi} * \left(\frac{2\pi N_{max}}{60} - \frac{9}{\sqrt{\frac{2mhk}{I_p}}}\right)$$

9. Conclusion

The dynamic mathematical model of the balancing robot is formulated by applying calculus to Newton's kinematic equation. A PID controller is designed to find out the difference between actual set balance point and out come from overall system. The measurement of accelerometer, analog to digital converter, PID controller and microcontroller generated PWM for actuators are used to balance a system and leads the whole system to balance point by rotating the motors either in clockwise or in anticlockwise direction. Measurement result shows stable and balanced points even for sever conditions.

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