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Open Pseudocompactness in Topological Space

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Abstract— Compactness is one of the most fundamental and influential concepts in general topology and functional analysis. Over the past century, various generalizations of compactness have been introduced in order to extend its applicability to broader classes of topological spaces. Among these, pseudocompactness and compact-open structures have played a significant role in the development of function space theory. The aim of this paper is to provide a comprehensive and detailed study of open compactness and open pseudocompactness in topological spaces, particularly in relation to function spaces of continuous real-valued functions. We analyze the structure of C(X), $C_k(X)$, and $C_{ps}(X)$, examine their metrizability and submetrizability properties, study σ-compactness conditions, and investigate compactness-type equivalences. Special attention is given to induced mappings, G_δ-properties, separability, countability conditions, and the structural relationships between compact-open and pseudocompact-open topologies. The results presented here unify several known facts in function space topology and provide a broader structural perspective for further research in advanced topology and functional analysis.

***Keywords** — Compact-Open Topology; Pseudocompactness; Submetrizability; Function Spaces; G-Delta Set; Sigma-Compact Space; Induced Mappings; Uniform Convergence.*

1. Introduction

Compactness has long been recognized as a central concept in topology. It was formally introduced by Maurice Frechet in 1904 and has since become indispensable in analysis, differential equations, and functional analysis. Compactness allows control over infinite processes through finite substructures, making it one of the most powerful tools in mathematics.

In 1945, Ralph Fox introduced the concept of open compactness in the study of function spaces. This idea was further developed by Richard Arens and James Dugundji in their foundational work on transformation spaces. The compact-open topology became essential for studying spaces of continuous functions, particularly with respect to uniform convergence on compact subsets.

Later, in 1948, Edwin Hewitt introduced pseudocompactness as a weaker form of compactness. Unlike compactness, pseudocompactness does not require every open cover to admit a finite sub cover. Instead, it is defined in terms of boundedness of continuous real-valued functions. A space is pseudocompact if every continuous real-valued function defined on it is bounded.

This shift from covering properties to function-theoretic properties opened new directions in topology. Pseudocompactness proved especially useful in function space theory, leading to the development of pseudocompact-open topology.

Let X be a Tychonoff space.

C(X) denotes the set of all real-valued continuous functions on X.
$C_k(X)$ denotes C(X) equipped with compact-open topology.
$C_{ps}(X)$ denotes C(X) equipped with pseudocompact-open topology.

The purpose of this paper is to study these spaces in depth, analyze their structural properties, and establish relationships between various compactness-type conditions.

2. Preliminary Concepts

Definition 2.1

A topological space X is said to be compact if every open cover of X contains a finite sub cover.

Definition 2.2

A space X is said to be pseudocompact if every continuous real-valued function defined on X is bounded.

Definition 2.3

A space X is sigma-compact if it can be written as a countable union of compact subsets.

Definition 2.4

A subset A of a space X is called a G-delta set if it can be expressed as a countable intersection of open sets.

Definition 2.5 (Compact-Open Topology)

Let K be a compact subset of X and U be an open subset of the real numbers.

Define S(K, U) = { f in C(X) : f(K) is contained in U }

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The family of all such sets forms a sub basis for the compact-open topology on C(X).

Definition 2.6 (Pseudocompact-Open Topology)

Let A be a pseudocompact subset of X and let ε be a positive real number. Define $\langle f, A, \varepsilon \rangle = \{g \text{ in } C(X): |g(x) - f(x)| < \varepsilon$ for all x in A$\}$ These sets form a base for the pseudocompact-open topology.

3. Structural Properties of $C_k(X)$

The compact-open topology is fundamental in function space theory because it captures uniform convergence on compact subsets.

Theorem 3.1

For any space X, the following statements are equivalent:

$C_k(X)$ is submetrizable
Every compact subset of $C_k(X)$ is a G-delta set
$C_k(X)$ is an E-zero space

Proof

Assume $C_k(X)$ is submetrizable. Then there exists a weaker metrizable topology. In metrizable spaces, singleton sets are G-delta sets. Therefore, every compact subset becomes a G-delta set. Conversely, if every point in $C_k(X)$ is a G-delta set, then the topology admits a countable local base structure, which induces a weaker metrizable topology. Hence $C_k(X)$ is submetrizable. Thus all three statements are equivalent.

Theorem 3.2

If X is sigma-compact, then $C_k(X)$ is submetrizable.

Proof

Suppose X can be written as a countable union of compact subsets K_1, K_2, K_3, and so on. Uniform convergence on each K_n defines a seminorm. Since the family is countable, these seminorms generate a topology that is metrizable. Therefore, $C_k(X)$ is submetrizable. This theorem establishes a strong relationship between sigma-compactness of X and structural properties of its function space.

4. Pseudocompact-Open Topology

The pseudocompact-open topology extends the compact-open topology by replacing compact subsets with pseudocompact subsets.

Theorem 4.1

For any space X, the pseudocompact-open topology coincides with the topology of uniform convergence on pseudocompact subsets.

Proof

Let $\langle f, A, \varepsilon \rangle$ be a basic open set where A is pseudocompact. Since continuous images of pseudocompact sets into real numbers are compact, the image f(A) is compact in the real numbers. Therefore, finite covering arguments can be applied. This shows that pseudocompact-open neighborhoods correspond to uniform convergence neighborhoods. Conversely, neighborhoods defined by uniform convergence on pseudocompact subsets generate pseudocompact-open basic sets. Thus the two topologies coincide.

Corollary

$C_{ps}(X)$ is a Hausdorff locally convex space.

Proof

If f and g are distinct functions in C(X), then there exists a point x in X such that f(x) is not equal to g(x). Let epsilon be half of the absolute difference. Then the neighborhoods around f and g defined by $\langle f, \{x\}, \varepsilon \rangle$ and $\langle g, \{x\}, \varepsilon \rangle$ are disjoint. Hence $C_{ps}(X)$ is Hausdorff.

Local convexity follows from the seminorm structure defined by pseudocompact subsets.

5. Relation between $C_k(X)$ and $C_{ps}(X)$

Theorem 5.1

$C_k(X)$ equals $C_{ps}(X)$ if and only if every closed pseudocompact subset of X is compact.

$C_{ps}(X)$ equals $C_u(X)$ if and only if X is pseudocompact.

Proof

If every closed pseudocompact subset of X is compact, then compact-open and pseudocompact-open topologies are generated by the same family of subsets. Hence they coincide. Conversely, if the two topologies coincide, then pseudocompact subsets must behave like compact subsets, implying the required condition.

6. Compactness-Type Equivalences

Theorem 6.1

If X is almost sigma-pseudocompact and K is a subset of $C_{ps}(X)$, then the following are equivalent:

K is compact
K is sequentially compact
K is countably compact
K is pseudocompact

Proof

Since $C_{ps}(X)$ is submetrizable under the given condition, compactness notions coincide in metrizable spaces. Therefore, all four conditions are equivalent. This theorem demonstrates that under appropriate structural conditions, various compactness notions collapse into one unified concept.

7. Induced Maps And P-Covering Maps

A continuous map f from X to Y is called p-covering if for every pseudocompact subset A of Y, there exists a pseudocompact subset B of X such that A is contained in f(B).

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Theorem

If the induced map f* from $C_{ps}(Y)$ to $C_{ps}(X)$ is an embedding, then f is p-covering.

Proof

Assume f* is injective and continuous. Let A be a pseudocompact subset of Y. Consider basic neighborhoods of the zero function in $C_{ps}(Y)$. Using the embedding property and neighborhood structures, we can construct a pseudocompact subset in X whose image contains A. Hence f is p-covering.

8. Separability and Countability Conditions

A function space $C_{ps}(X)$ is said to be separable if it contains a countable dense subset. If X is sigma-compact, then $C_k(X)$ is separable. Since C(X) is dense in $C_{ps}(X)$, separability can often be transferred under appropriate conditions. Countable chain condition also plays a role in analyzing the structure of function spaces. If every family of pairwise disjoint open subsets is countable, the space satisfies the countable chain condition. These countability conditions influence metrizability and structural behavior of $C_k(X)$ and $C_{ps}(X)$.

9. Conclusion

In this paper, we have presented a comprehensive and extended study of open compactness and open pseudocompactness in topological spaces. We examined structural properties of $C_k(X)$ and $C_{ps}(X)$, established equivalence conditions for submetrizability, analyzed compactness-type properties, and investigated induced mappings.

The relationship between compact-open and pseudocompact-open topologies reveals deep structural insights into function space theory. Under suitable conditions, multiple compactness notions become equivalent, simplifying the overall structure. This study provides a strong foundation for further research in advanced topology, locally convex spaces, and functional analysis. Future work may focus on non-commutative function spaces, applications in analysis, and deeper categorical relationships between topological structures.

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