

A Study on Fibonacci Numbers and the Golden Ratio

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Abstract— Fibonacci numbers and the golden ratio are fundamental concepts in number theory and mathematics. The Fibonacci sequence is defined by a recursive relation in which each term is the sum of the two preceding terms. The golden ratio is a mathematical constant closely related to the Fibonacci sequence. This paper presents definitions, properties, identities, and important results related to Fibonacci numbers and the golden ratio. Important identities such as the sum of Fibonacci numbers, sum of squares, Fibonacci Q-matrix, Cassini's identity, and Binet's formula are discussed. The relationship between Fibonacci numbers and the golden ratio is analyzed. Applications in nature, computer science, and mathematics are also presented.

Keywords — Fibonacci numbers; golden ratio; Fibonacci sequence; Q-matrix, Binet's formula.

1. Introduction

Fibonacci numbers are an important sequence in number theory and mathematics. They were introduced by the Italian mathematician Leonardo of Pisa, also known as Fibonacci, in his book Liber Abaci (1202). The Fibonacci sequence is defined as:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Each number is the sum of the two preceding numbers. Fibonacci numbers have applications in mathematics, computer science, biology, and art. They are closely related to the golden ratio.

The golden ratio is defined as: $\Phi = (1 + \sqrt{5}) / 2 \approx 1.618$. The ratio of consecutive Fibonacci numbers approaches the golden ratio.

2. Definitions and Basic Properties

Definition: Fibonacci Sequence

The Fibonacci series is a sequence of numbers where each number is the sum of the two preceding ones, typically starting with 0 and 1, creating the pattern:

0, 1, 1, 2, 3, 5, 8, 13, 21, and so on

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}$$

with initial values:

$$F_1 = 1, F_2 = 1$$

Definition: Golden Ratio

The golden ratio is defined as the positive solution of:

$$\Phi^2 - \Phi - 1 = 0$$

Solving,

$$\Phi = (1 + \sqrt{5})/2$$

Limit of Fibonacci Ratio

The ratio of consecutive Fibonacci numbers approaches the golden ratio.

$$F_{n+1} / F_n = 1 + F_{n-1} / F_n$$

Taking limit as $n \rightarrow \infty$,

Let limit = α

$$\text{Then, } \alpha = 1 + 1/\alpha$$

$$\text{Solving, } \alpha^2 - \alpha - 1 = 0$$

$$\text{Therefore, } \alpha = \Phi$$

3. Sum of Fibonacci Numbers

I derive the summation identity

$$\sum_{i=1}^n F_i = F_{n+2} - 1. \tag{1.1}$$

For example, consider the first eight Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21. With $n = 6$ in (1.1), we have

$$\sum_{i=1}^6 F_i = 1 + 1 + 2 + 3 + 5 + 8 = 20, \text{ and}$$

$$F_{n+2} - 1 = 21 - 1 = 20.$$

One can use mathematical induction to prove (1.1), but a direct derivation uses the relation $F_n = F_{n+2} - F_{n+1}$

Constructing a list of identities, we have

$$F_n = F_{n+2} - F_{n+1}$$

$$F_{n-1} = F_{n+1} - F_n$$

$$F_{n-2} = F_n - F_{n-1} \dots \dots \dots$$

$$F_2 = F_4 - F_3$$

$$F_1 = F_3 - F_2.$$

Adding all the left-hand sides yields the sum over the first n Fibonacci numbers and adding all the right-hand side results in the cancellation of all terms except the first and the last.

4. Sum of Fibonacci Squares

I derive a combinatorial identity obtained by summing over the squares of the Fibonacci numbers:

$$\sum_{i=1}^n F_{2i} = F_n F_{n+1}. \tag{1.2}$$

For example, consider the first seven Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13. With $n = 6$ in (1.2), we have

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8 \times 13,$$

where by doing the arithmetic one finds that both sides are equal to 104. we work with the right-hand side, using the Fibonacci recursion relation. We have

$$\begin{aligned} F_n F_{n+1} &= F_n (F_n + F_{n-1}) \\ &= F_n^2 + F_{n-1} F_n \end{aligned}$$

$$= F_n^2 + F_{n-1} (F_{n-1} + F_{n-2})$$

$$= F_n^2 + F_{n-1}^2 + F_{n-2} F_{n-1}$$

= ...

$$= F_n^2 + F_{n-1}^2 + \dots + F_2^2 + F_1 F_2$$

Because $F_2 = F_1$, the identity (1.2) is proved.

5. Cassini's Identity

The Fibonacci Q-matrix is given by

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

$$\text{With } Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

From the theory of matrices and determinants we know that $\det AB = \det A \det B$.

Repeated application of this result yields $\det Q^n = (\det Q)^n$

Applying this results directly in Cassini's identity,

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$$

Examples of this equality can be obtained from the first few numbers of the Fibonacci Sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... We have

$$2 \times 5 - 3^2 = 1$$

$$3 \times 8 - 5^2 = -1$$

$$5 \times 13 - 8^2 = 1$$

$$8 \times 21 - 13^2 = -1$$

$$13 \times 34 - 21^2 = 1.$$

Cassini's identity is the basis of an amusing dissection fallacy, called the Fibonacci bamboozlement.

6. Golden Spiral

The celebrated golden spiral is a special case of the more general logarithmic spiral whose radius r is given by $r = e^{b\theta}$, where θ is the usual polar angle, and a and b are constants. The golden spiral is a logarithmic spiral whose radius either increases or decreases by a factor of the golden ratio Φ with each one-quarter turn, that is, when θ increases by $\pi/2$. The golden spiral therefore satisfies the equation $r = \Phi^{2\theta/\pi}$.

7. Golden Angle

The golden angle is defined as the acute angle g that divides the circumference of a circle into two arcs with lengths in the golden ratio. The golden ratio Φ and the golden ratio conjugate ϕ satisfy

$$\Phi = \frac{x}{y}, \phi = \frac{y}{x},$$

with $\Phi = 1 + \phi$. We can determine the golden angle by writing

$$\frac{g}{2\pi} = \frac{y}{x+y} = \frac{\phi}{1+\phi} = \frac{\phi}{\Phi} = \phi^2;$$

And since $\phi^2 = 1 - \phi$, we obtain

$$g = 2\pi(1 - \phi).$$

To determine the continued fraction for $g/2\pi$, we write

$$\begin{aligned} \frac{g}{2\pi} &= \frac{y}{x+y} = \frac{1}{1+\Phi} \\ &= \frac{1}{1+1+\frac{1}{1+\frac{1}{1+\dots}}} \end{aligned}$$

which yields $g/2\pi = [0; 2, 1]$.

8. Binet's Formula

The Fibonacci numbers are uniquely determined from their recursion relation,

$$F_{n+1} = F_n + F_{n-1}, \tag{1.3}$$

and the initial values, $F_1 = F_2 = 1$. An explicit formula for the Fibonacci numbers can be found, and is called Binet's Formula. To solve (1.3) for the Fibonacci numbers, we first look at the equation

$$x_{n+1} = x_n + x_{n-1}. \tag{1.4}$$

This equation is called a second-order, linear, homogeneous difference equation with constant coefficients, and its method of solution closely follows that of the analogous differential equation. The idea is to guess the general form of a solution, find two such solutions, and then multiply these solutions by unknown constants and add them. This results in a general solution to (1.4), and one can then solve (1.3) by satisfying the specified initial values.

To begin, we guess the form of the solution to (1.4) as

$$x_n = \lambda, \tag{1.5}$$

where λ is an unknown constant. Substitution of this guess into (1.4) results in

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1},$$

or upon division by λ^{n-1} and rearrangement of terms,

$$\lambda^2 - \lambda - 1 = 0.$$

Use of the quadratic formula yields two roots, both of which are already familiar. We have

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \Phi, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} = -\phi,$$

where Φ is the golden ratio and ϕ is the golden ratio conjugate.

We have thus found two independent solutions to (1.4) of the form (1.5), and we can now use these two solutions to find a solution to (1.3). Multiplying the solutions by constants and adding them, we obtain

$$F_n = c_1 \Phi^n + c_2 (-\phi)^n, \tag{1.6}$$

which must satisfy the initial values $F_1 = 1$ and $F_2 = 1$. The algebra for finding the unknown constants can be made

simpler, however, if instead of F_2 , we use the value $F_0 = F_2 - F_1 = 0$.

Application of the values for F_0 and F_1 results in the system of equations given by

$$\begin{aligned}c_1 + c_2 &= 0, \\c_1\Phi - c_2\varphi &= 1.\end{aligned}$$

We use the first equation to write $c_2 = -c_1$, and substitute into the second equation to get

$$\begin{aligned}c_1(\Phi + \varphi) &= 1. \\ \text{Since } \Phi + \varphi &= \sqrt{5}, \text{ we can solve for } c_1 \text{ and } c_2 \text{ to obtain} \\ c_1 &= 1/\sqrt{5}, c_2 = -1/\sqrt{5}.\end{aligned}\tag{1.7}$$

Using (1.7) in (1.6) then derives the surprising formula

$$F_n = \frac{\Phi^n - (-\varphi)^n}{\sqrt{5}},$$

known as Binet's formula

9. Applications

The Fibonacci sequence and golden ratio have numerous applications in nature, art, design, finance, biology, and computer science. They describe patterns in phyllotaxis, architecture, and technical analysis, showcasing their significance in understanding structures.

10. Conclusion

Fibonacci numbers and the golden ratio are closely related mathematical concepts. Important identities such as sum formula, Cassini's identity, and Binet's formula were discussed. The golden ratio appears naturally in Fibonacci numbers. These concepts have applications in mathematics, science, and engineering.

References

- [1] Jeffrey R. Chasnov, *Fibonacci Numbers and Golden Ratio*, Hong Kong University of Science and Technology (HKUST), 2015.
- [2] Richard A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Publishing, 1997.
- [3] Keith Devlin, *The Man of Numbers: Fibonacci's Arithmetic Revolution*, Walker & Company, 2011.
- [4] Thomas Koshy, *Fibonacci and Lucas Numbers with Application*, John Wiley & Sons Ltd., Volume 2 (second edition series), 2019