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Insight on Ramanujan's Puzzle

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Abstract— One of the most significant mathematicians of India, of 20th century was Srinivasa Ramanujan. He posed several interesting puzzles as his notebook jottings. The entries of his notebooks were today considered to be mathematical treasures. During the period when Ramanujan stayed at Trinity College of Cambridge University, England his Indian friend posed an interesting puzzle which Ramanujan had solved immediately. In this paper, I will introduce that puzzle and provide the way as how Ramanujan could have accomplished the solution in a flash.

Keywords ***— House Number Puzzle; Pell's Equation; Continued Fraction; Convergents.***

1. Introduction

P.C. Mahalanobis, a profound statistician from India who went to England for doing his research became close associate with Ramanujan. They both discuss several issues about life and mathematics. One day when Mahalanobis visited Ramanujan he mentioned about a puzzle which appeared in a magazine called The Strand, the previous day. When Ramanujan heard the puzzle, he could provide the solution in a flash. In this paper, I will describe the puzzle in the historical context as presented in [8] and provide a way of solving it, as was probably done by Ramanujan.

2. Description of the Puzzle

There are two versions of the puzzle that were been circulated in history of Ramanujan. I am describing a puzzle which is presented in the first biography of Ramanujan titled "Ramanujan: The man and the mathematician" (see [5]) written by Dr. S. R. Ranganathan who is accomplished mathematician himself and considered to be father of Library Science in India. The history about this puzzle as well as other puzzles related to Ramanujan can be viewed at [1-4, 6-7].

MAHALANOBIS: My friend, here is a puzzle for you
RAMANUJAN: What puzzle, tell me.
MAHALANOBIS: "Two British officers had been billeted in Paris in two different houses in a long street; the door numbers of these houses were related in a special way; the problem was to find out the two numbers. It was not at all difficult. I got the solution in a few minutes by trial and error.
RAMANUJAN: Please take down the solution. (He dictated a continued fraction).

The first term was the solution which Mahalanobis had obtained by trial and error method. But when Mahalanobis saw the solution by Ramanujan, he was taken to complete surprise, since it provided not only the solution to the given puzzle but also generated infinitely many solutions.

MAHALANOBIS: How did you arrive at this solution?
RAMANUJAN: The moment I noticed the puzzle, it was clear to my mind that the solution would be a continued fraction. Then I thought for a while to recognize what continued fraction would lead the solution. In few seconds, I recognized it and provided the same to you.

The following is one of the modified versions of the house number puzzle provided by Shri P. K. Srinivasan, who was chiefly responsible for highlighting Ramanujan's life and achievements to common man.

Two officers went for a walk along a long street. The following is their conversation which describes the puzzle.
First Officer: I noticed a special relationship existing among the house numbers that were allotted for us.
Second Officer: What is that?
First Officer: Ten times the square of your house number is always one more or one less than the square of my house number. Moreover, your house number is greater than 50 but not as much as 500. Can you determine your house number?

It is very clear from the description of the puzzle that the First Officer most probably might be the senior among the two and he already knew their house numbers and just posed this puzzle to check if his junior colleague can figure it out.

I will now provide the solution to this puzzle by the methods that Ramanujan could have adopted before informing his answer to Mahalanobis.

3. Solution to the Puzzle

If we assume that x and y are house numbers of the first and second officers respectively, then according to the description of the problem we obtain $x^2 - 10y^2 = \pm 1$ (1)
Equations of the form in (1) are usually called Pell's Equation, though the name Pell has nothing to do with such equations. In fact, ancient Indian mathematician

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Brahmagupta knew solutions to those equations and had provided a recursive formula which could generate infinitely many solutions in integers. But Ramanujan solved (1) using the concept of continued fraction, as can be seen by his reply to Mahalanobis in describing the puzzle in previous section.

So, what could be Ramanujan's solution? What continued fraction that came to Ramanujan's mind in providing the solution to Mahalanobis? In this paper, I will discuss the answers to these questions.
First, we realize that $(3-\sqrt{10})\times(3+\sqrt{10})=-1$ (2)
Using (2) we can generate the following computations

$$3-\sqrt{10}=\frac{-1}{3+\sqrt{10}}=\frac{-1}{6-(3-\sqrt{10})}=\frac{-1}{6+\cfrac{1}{6-(3-\sqrt{10})}}=\frac{-1}{6+\cfrac{1}{6+\cfrac{1}{6-(3-\sqrt{10})}}}$$

$$=\frac{-1}{6+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6-(3-\sqrt{10})}}}}=\cdots=\frac{-1}{6+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\cdots}}}}$$

Thus we arrive at the continued fraction for $\sqrt{10}$ as

$$\sqrt{10}=3+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\cdots}}}} \quad (3)$$

Now considering the successive convergent of the continued fraction in (3), we get

$$\frac{3}{1},3+\frac{1}{6}=\frac{19}{6},3+\cfrac{1}{6+\cfrac{1}{6}}=\frac{117}{37},3+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6}}}=\frac{721}{228},3+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6}}}}=\frac{4443}{1405},\ldots \quad (4)$$

The successive convergent obtained in (4) will provide the solutions for (1). In particular, the numerators of the above rational numbers represent the house numbers of first officer namely *x* and denominators will represent the house numbers of the second officer namely *y* respectively, Moreover, from (4), if we consider first, third, fifth, in general odd indexed convergent ratios then they represent solutions to the equation $x^2-10y^2=-1$ (5), whereas if we consider second, fourth, in general even indexed convergent ratios then they represent solutions to the equation $x^2-10y^2=1$ (6).

For example, we notice that $(x,y)=(3,1);(117,37);(4443,1405);\ldots$ are solutions to $x^2-10y^2=-1$ and $(x,y)=(19,6);(721,228);\ldots$ are solutions of $x^2-10y^2=1$.

It is sure that Mahalanobis recognized thefirst solutions x = 3, y = 1, but according to the requirement of puzzle, house number of second officer *y*must be greater than 50 but less than 500. Hence, among the successive solutions represented in (4), we notice that the required solution is $x=721, y=228$. That is, the house number of the second officer must be 228 and that of first officer must be 721.

Continuing to extract more convergents from the continued fraction (3), we can get infinitely many solutions to the puzzle. Thus Ramanujan not only solved the problem with the required solution but his continued fraction for $\sqrt{10}$ paved way for generating infinitely many solutions at one stroke. This single example is enough to recognize Ramanujan's genius thinking in mathematics. Infinitely many solutions in integers. But Ramanujan solved (1) using the concept of continued fraction, as can be seen by his reply to Mahalanobis in describing the puzzle in previous section.

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For example, we notice that $(x, y) = (3,1); (117,37); (4443,1405); ...$ are solutions to $x^2 - 10y^2 = -1$ and $(x, y) = (19,6); (721,228); ...$ are solutions of $x^2 - 10y^2 = 1$.

It is sure that Mahalanobis recognized thefirst solutions $x = 3$, $y = 1$, but according to the requirement of puzzle, house number of second officer $y$must be greater than 50 but less than 500. Hence, among the successive solutions represented in (4), we notice that the required solution is $x = 721, y = 228$. That is, the house number of the second officer must be 228 and that of first officer must be 721.

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4. Conclusion

In this paper, I have introduced the famous house number puzzle in connection with the conversation between Mahalanobis and Ramanujan. It was very clear that Ramanujan provided the solution using a continued fraction in very short time. This is possible, because Ramanujan immediately recognized that the posed puzzle ultimately reduces to solving Pell's equation as given in (1). He also knew from his notebook jottings that the continued fraction corresponding to $\sqrt{10}$ was the one obtained in (3) in this paper.

Hence, Ramanujan could have thought the continued fraction for $\sqrt{10}$ as in (3) and obtained not just the required solution but also could generate infinitely many solutions. This single incident is more than enough to show how fast Ramanujan's mind was in connecting mathematical concepts, which was the major factor for him to produce more than 3000 outstanding formulas in his short life span of about 32 years. I am quite excited to read Ramanujan's mind in the way probably he could have thought in solving this puzzle. I dedicate this paper to Ramanujan commemorating his 134th birth anniversary.

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