Engineering and Scientific International Journal (ESIJ)
Volume 8, Issue 3, July – September 2021

ISSN 2394-7187(Online)
ISSN 2394 - 7179 (Print)

Analysis of a Parabolic Fin via Matrix Method

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Abstract— Heat is not lost equally by each element of the fin but is lost mostly near the base of the fin. Thus there would be wastage of the material if a uniform fin is used. Due to this reason fins of varying cross-sections like hyperbolic fins or parabolic fins are constructed. The parabolic fin of varying cross-section is usually analyzed by ordinary calculus approach. The paper analyses parabolic fin of varying cross-section to find the rate of conduction of heat through it via the application of matrix method. The matrix method has been applied successfully in science and engineering problems and also comes out to be very effective tool to find the temperature distribution and rate of conduction of heat through a parabolic fin.

Keywords — *Parabolic Fin; Temperature Distribution; Matrix Method.*

1. Introduction

The conduction of heat takes place through the fins from particle to particle due to temperature gradient in the direction of decreasing temperature [1-3]. Fins are the extended surfaces which are mostly used in the devices which exchange heat [4-9] like computer central processing unit, power plants, radiators, heat sinks, etc. The matrix method has been applied successfully in science and engineering problems [10-18]. The paper puts forward the Matrix Method for the analysis of a parabolic fin of varying cross-section which is usually analyzed by ordinary calculus approach [4-9]. The paper analyzes parabolic fin of varying cross-section to find the rate of conduction of heat through it via the application of matrix method. The Matrix method also comes out to be very effective tool to find the temperature distribution and rate of conduction of heat through a parabolic fin. As we know, for a square matrix B [11-14]of order n with elements a_{ij}, we get a column matrix Z and a constant λ such that BZ = λZ or $|B - \lambda I|$Z = 0. This represents a matrix equation which results in n homogeneous linear equations [12-15] having a non–trivial solution only if $|B - \lambda I|$= 0. On expand $|B - \lambda I|$, we will get n[th] degree equation in λ, known as the characteristic equation of B, whose roots i.e. λ_i $(where\ i = 1, 2, 3, \ldots\ldots n)$ are called eigen values and corresponding to each eigen value there is a non-zero solution Z = $\begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_n \end{bmatrix}$ known as eigen vector [13-17].

2. Materials and Method

The differential equation for analyzing a parabolic fin [4-9] (assuming that heat flow pertains to one dimensional conduction of heat) is given by,

$$x^2\ddot{\theta}(x) + 2x\dot{\theta}(x) - m^2 l^2\theta(x) = 0 \ldots\ldots (1),$$

$where\ m = \sqrt{\frac{2h}{tk}}$, l is the length of the fin between the base at x = l and the tip at x = 0, t is the thickness of the fin which increases uniformly from zero at the tip to t at the base, k is thermal conductivity, h is the coefficient of transfer of heat by convection, $\theta(x) = T(x) - T_s$, T_s is the temperature of the environment of the fin and T_0 is the temperature at the base x = 0 of the fin.

Substituting $x = e^z$, the equation (1) can be rewritten into a form:

$$\ddot{\theta}(x) + \dot{\theta}(x) - m^2 l^2\theta(x) = 0 \ldots\ldots (2)$$

$$\cdot \equiv \frac{d}{dz}$$

Further, let us substitute

$$\theta\ (x) = \theta_1(x) \ldots\ldots\ldots . (3)$$

and

$$\dot{\theta}_1(x) = \theta_2(x) \ldots\ldots. (4)$$

We can rewrite equation (2) as

$$\dot{\theta}_2(x) + \theta_2(x) - m^2L^2\theta_1(x) = 0$$

Or

$$\dot{\theta}_2(x) = m^2L^2\theta_1(x) - \theta_2(x) \ldots . (5)$$

We can write the equations (4) and (5) in the matrix form as

$$\begin{bmatrix} \dot{\theta}_1(x) \\ \dot{\theta}_2(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ m^2L^2 & -1 \end{bmatrix} \begin{bmatrix} \theta_1(x) \\ \theta_2(x) \end{bmatrix}$$

On equating the determinant of $\begin{bmatrix} 0 & 1 \\ m^2L^2 & -1 \end{bmatrix}$ to zero, we can obtain its characteristic equation as

$$\begin{vmatrix} 0 - \mu & 1 \\ m^2L^2 & -1 - \mu \end{vmatrix} = 0$$

On expanding the determinant, we get

$$\mu^2 + \mu - m^2L^2 = 0 \ldots\ldots. (6)$$

This equation (6) is quadratic in μ and its roots are given by

DOI: **10.30726/esij/v8.i3.2021.83016**

ISSN 2394-7187(Online)
ISSN 2394 - 7179 (Print)

$$\mu = \frac{-1 \pm \sqrt{(1)^2 + 4m^2L^2}}{2}$$

Or

$$\mu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + m^2L^2}$$

The roots of the equation (6) are

$$\mu_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + m^2L^2} \quad \ldots\ldots. (7)$$

And

$$\mu_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} + m^2L^2} \ldots\ldots\ldots.. (8)$$

The product of roots is given by

$$\mu_1\mu_2 = \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + m^2L^2}\right)\left(-\frac{1}{2} - \sqrt{\frac{1}{4} + m^2L^2}\right)$$

On simplifying the right hand side of this equation, we get

$$\mu_1\mu_2 = -m^2L^2 \ldots\ldots. (9)$$

Also, the sum of roots is given by

$$\mu_1 + \mu_2 = \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + m^2L^2}\right) + \left(-\frac{1}{2} - \sqrt{\frac{1}{4} + m^2L^2}\right)$$

Or

$$\mu_1 + \mu_2 = -1 \ldots\ldots. (10)$$

To determine Eigenvectors:

The Eigenvector corresponding to the root $\mu = \mu_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + m^2L^2}$ is given by

$$\begin{bmatrix} 0-\mu_1 & 1 \\ m^2L^2 & -1-\mu_1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This results,

$$-\mu_1 z_1 + z_2 = 0 \ldots\ldots.. (11)$$

And

$$m^2L^2 z_1 - (1+\mu_1)z_2 = 0 \ldots\ldots\ldots.. (12)$$

Solving equations (11) and (12), we can write

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \mu_1 + 2 \\ \mu_1 + m^2L^2 \end{bmatrix}$$

And the Eigenvector for the root $\mu = \mu_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} + m^2L^2}$ is given by

$$\begin{bmatrix} 0-\mu_2 & 1 \\ m^2L^2 & -1-\mu_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This results

$$-\mu_2 z_1 + z_2 = 0 \ldots\ldots.. (13)$$

And

$$m^2L^2 z_1 - (1+\mu_2)z_2 = 0 \ldots\ldots\ldots.. (14)$$

Solving equations (13) and (14), we can write

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \mu_2 + 2 \\ \mu_2 + m^2L^2 \end{bmatrix}$$

The matrix of Eigenvectors is $\begin{bmatrix} \mu_1 + 2 & \mu_2 + 2 \\ \mu_1 + m^2L^2 & \mu_2 + m^2L^2 \end{bmatrix}$.

Let $P = \begin{bmatrix} \mu_1 + 2 & \mu_2 + 2 \\ \mu_1 + m^2L^2 & \mu_2 + m^2L^2 \end{bmatrix}$, then the determinant of P i.e. $|P|$ is given by

$$|P| = \begin{vmatrix} \mu_1 + 2 & \mu_2 + 2 \\ \mu_1 + m^2L^2 & \mu_2 + m^2L^2 \end{vmatrix}$$

On expanding the determinant and making use of equation (10), we get

$$|P| = (\mu_2 - \mu_1)(-m^2L^2 + 2)$$

The inverse of P which is also a matrix is given by

$$P^{-1} = \frac{1}{(\mu_2 - \mu_1)(-m^2L^2 + 2)}\begin{bmatrix} \mu_2 + m^2L^2 & -(\mu_2 + 2) \\ -(\mu_1 + m^2L^2) & \mu_1 + 2 \end{bmatrix}$$

To Determine $\mathbf{Pe^{\mu z}P^{-1}}$:

$$Pe^{\mu z}P^{-1} = \begin{bmatrix} M_1 + 2 & \mu_2 + 2 \\ \mu_1 + m^2L^2 & \mu_2 + m^2L^2 \end{bmatrix}\begin{bmatrix} \exp(\mu_1 z) & 0 \\ 0 & \exp(\mu_2 z) \end{bmatrix} \times \frac{1}{(\mu_2 - \mu_1)(-m^2L^2 + 2)}\begin{bmatrix} \mu_2 + m^2L^2 & -(\mu_2 + 2) \\ -(\mu_1 + m^2L^2) & \mu_1 + 2 \end{bmatrix}$$

$$= \frac{1}{(\mu_2 - \mu_1)(\frac{K}{m} + \frac{r}{m} + 1)} \times \begin{bmatrix} (\mu_1 + 2)\exp(\mu_1 z) & (\mu_2 + 2)\exp(\mu_2 z) \\ (\mu_1 + m^2L^2)\exp(\mu_1 z) & (\mu_2 + m^2L^2)\exp(\mu_2 z) \end{bmatrix}\begin{bmatrix} \mu_2 + m^2L^2 & -(\mu_2 + 2) \\ -(\mu_1 + m^2L^2) & \mu_1 + 2 \end{bmatrix}$$

$$= \frac{1}{(\mu_2 - \mu_1)(-m^2L^2 + 2)} \times \begin{bmatrix} (\mu_1 + 2)(\mu_2 + m^2L^2)\times \exp(\mu_1 z) - (\mu_1 + m^2L^2)\times (\mu_2 + 2)\exp(\mu_2 z) & (\mu_1 + 2)(\mu_2 + 2)\times [\exp(\mu_2 z) - \exp(\mu_1 z)] \\ (\mu_2 + m^2L^2)(\mu_1 + m^2L^2)\times [\exp(\mu_1 z) - \exp(\mu_2 z)] & -(\mu_2 + 2)(\mu_1 + m^2L^2)\times \exp(\mu_1 z) + (\mu_1 + 2)\times (\mu_2 + m^2L^2)\exp(\mu_2 z) \end{bmatrix}$$

Now applying the initial boundary conditions: $\theta_1(0) = \theta(0) = C$ and $\theta_2(0) = \dot{\theta}(0) = D$ provides

$$\begin{bmatrix} \theta_1(x) \\ \theta_2(x) \end{bmatrix} = \frac{1}{(\mu_2 - \mu_1)(-m^2L^2 + 2)} \times \begin{bmatrix} (\mu_1 + 2)(\mu_2 + m^2L^2)\times \exp(\mu_1 z) - (\mu_1 + m^2L^2)\times (\mu_2 + 2)\exp(\mu_2 z) & (\mu_1 + 2)(\mu_2 + 2)\times [\exp(\mu_2 z) - \exp(\mu_1 z)] \\ (\mu_2 + m^2L^2)(\mu_1 + m^2L^2)\times [\exp(\mu_1 z) - \exp(\mu_2 z)] & -(\mu_2 + 2)(\mu_1 + m^2L^2)\times \exp(\mu_1 z) + (\mu_1 + 2)\times (\mu_2 + m^2L^2)\exp(\mu_2 z) \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}$$

This results,

$$\theta_1(x) = \frac{C[(\mu_1 + 2)(\mu_2 + m^2L^2)\exp(\mu_1 z) - (\mu_1 + m^2L^2)(\mu_2 + 2)\exp(\mu_2 z)] + D(\mu_1 + 2)(\mu_2 + 2)[\exp(\mu_2 z) - \exp(\mu_1 z)]}{(\mu_2 - \mu_1)(-m^2L^2 + 2)}$$

Or

DOI: **10.30726/esij/v8.i3.2021.83016**

ISSN 2394-7187(Online)
ISSN 2394 - 7179 (Print)

$$\theta(t) = \frac{C[(\mu_1+2)(\mu_2+m^2L^2)\exp(\mu_1 z) - (\mu_1+m^2L^2)(\mu_2+2)\exp(\mu_2 z)] + D(\mu_1+2)(\mu_2+2)[\exp(\mu_2 z)-\exp(\mu_1 z)]}{(\mu_2-\mu_1)(-m^2L^2+2)} \ldots (15)$$

Now making use of equations (9) and (10), we can find that

$(\mu_1 + 2)(\mu_2 + m^2L^2) = \mu_2(-m^2L^2 + 2)$ …. (16)

And $(\mu_1 + m^2L^2)(\mu_2 + 2) = \mu_1(-m^2L^2 + 2)$ …….. (17)

$(\mu_1 + 2)(\mu_2 + 2) = (-m^2L^2 + 2)$…. (18)

Put equations (16), (17) and (18) in equation (15), we obtain

$$\theta(x) = \frac{C[\mu_2 \exp(\mu_1 z) - \mu_1 \exp(\mu_2 z)] + D[\exp(\mu_2 z) - \exp(\mu_1 z)]}{\mu_2 - \mu_1}$$

Or

$$\theta(x) = \frac{C[\mu_1 \exp(\mu_2 z) - \mu_2 \exp(\mu_1 z)] + D[\exp(\mu_2 z) - \exp(\mu_1 z)}{\mu_1 - \mu_2}$$

Or

$$\theta(x) = \frac{C[\mu_1 \exp(\mu_2 z) - \mu_2 \exp(\mu_1 z)] + D[\exp(\mu_2 z) - \exp(\mu_1 z)}{\mu_1 - \mu_2}$$

Or

$$\theta(x) = \frac{[(\mu_1)C + D]e^{\mu_1 z} + [-(\mu_2)C - D]e^{-\mu_2 z}}{\mu_1 - \mu_2}$$

Or

$$\theta(x) = \frac{[(\mu_1)C + D]x^{\mu_1} + [-(\mu_2)C - D]x^{-\mu_2}}{\mu_1 - \mu_2} \ldots (19)$$

As $\theta(0)$is finite [4-9], therefore, the term $[-(\mu_2)C - D]x^{-\mu_2}$ *is equated to zero*.
$i.e. [-(\mu_2)C - D]x^{-\mu_2} = 0$, *which gives*
$D = -(\mu_2)C$

From (19), we have

$\theta(x) = C(\mu_1 - \mu_2)\, x^{\mu_1}$ …. (20)

To find the constant C, at x = l,$\theta(l) = \theta_0$,[4-9] therefore, from (20)

$C = \frac{1}{\mu_1 - \mu_2}\theta_0 l^{-\mu_1}$

Hence (20) can be rewritten as

$\theta(x) = \theta_0 l^{-\mu_1} x^{\mu_1}$

Or

$\theta(x) = \theta_0 (x/l)^{\mu_1}$… (21)

The equation (21) gives the temperature distribution along the length of the parabolic fin.

The heat conducted through the fin is given by the Fourier's Law of heat conduction [5, 19, 20, 21,] as

$H = kA\left(\theta'(x)\right)_{x=l} = kbt\left(\theta'(x)\right)_{x=l}$

Using equation (21), we get

$H = kbt\, \theta_0 \mu_1 / l$

Or

$H = kbt\, \theta_0 \frac{-1+(1+4M^2 l^2)^{1/2}}{2l}$ ….. (22)

This equation (22) gives the rate of conduction of heat through the parabolic fin.

The heat conducted through the fin is given by the Fourier's Law of heat conduction [5, 19, 20, 21,] as

$H = kA\left(\theta'(x)\right)_{x=l} = kbt\left(\theta'(x)\right)_{x=l}$

Using equation (21), we get

$H = kbt\, \theta_0 \mu_1 / l$

Or

$H = kbt\, \theta_0 \frac{-1+(1+4M^2 l^2)^{1/2}}{2l}$ ….. (22)

This equation (22) gives the rate of conduction of heat through the parabolic fin.

3. Result

It has been found that the temperature of parabolic fin increases with the increase in its length. Also,, the rate of conduction of heat at any cross-section of the parabolic fin increases with the increase in its length. The matrix method has been applied successfully for the analysis of a parabolic fin to find the temperature distribution along its length, and the rate of conduction of heat through it. The results obtained are the same as obtained with other methods or approaches [4-9].

4. Conclusion

The rate of heat transfer in a parabolic fin is calculated by solving simple differential equation by the application of matrix method. The matrix method approach has come out to be a very effective tool for the analysis of a parabolic fin to find the temperature distribution along its length and the rate of conduction of heat through it.

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DOI: **10.30726/esij/v8.i3.2021.83016**